Turbulente Strömungen
WS 2015/2016
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Handout #4

Handout # 4:
Equations of Fluid Motion

Substantial derivative (describes change of fluid particle moving with local flow velocity)

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + U_i \frac{\partial}{\partial x_i}$$

Continuity Equation

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho U_i) = 0$$

Incompressibility condition

$$\frac{D\rho}{Dt} \equiv 0 \quad \Rightarrow \quad \frac{\partial \rho}{\partial t} + U_i \frac{\partial \rho}{\partial x_i} = \frac{\partial \rho}{\partial t} + \frac{\partial (\rho U_i)}{\partial x_i} - \rho \frac{\partial U_i}{\partial x_i} = 0$$

By continuity equation, this gives

$$\frac{\partial U_i}{\partial x_i} = 0$$

Example: Two immiscible fluids of different density

Incompressible flows are divergence free or solenoidal

Momentum Equations

$$\rho \frac{DU_j}{Dt} = \rho \frac{\partial U_j}{\partial t} + \rho U_i \frac{\partial U_j}{\partial x_i} = \frac{\partial \tau_{ij}}{\partial x_i} \quad \text{(neglecting body forces)}$$

The stress tensor τ_{ij} for constant property Newtonian fluid given by

$$\tau_{ij} = -P\delta_{ij} + \mu\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right)$$

$$\frac{\partial U_j}{\partial t} + U_i\frac{\partial U_j}{\partial x_i} = -\delta_{ij}\frac{1}{\rho}\frac{\partial P}{\partial x_i} + \nu\frac{\partial}{\partial x_i}\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right)$$

From continuity equation: $\partial U_i/\partial x_i = 0$, this reduces to

$$\frac{\partial U_j}{\partial t} + U_i\frac{\partial U_j}{\partial x_i} = -\frac{1}{\rho}\frac{\partial P}{\partial x_j} + \nu\frac{\partial^2 U_j}{\partial x_i^2}, \qquad \nu = \frac{\mu}{\rho}$$

Special form for inviscid fluid: $\mu = 0$

$$\frac{DU_j}{Dt} = -\frac{1}{\rho}\frac{\partial P}{\partial x_j}$$

Consequence: Acceleration caused by pressure gradient

Questions:

1. Why do we need continuity equation?
2. Where do we get the pressure from?

Poisson Equation

Take the divergence of the momentum equation:

$$\frac{\partial}{\partial x_j}\left(\frac{\partial U_j}{\partial t} + U_i\frac{\partial U_j}{\partial x_i} = -\frac{1}{\rho}\frac{\partial P}{\partial x_j} + \nu\frac{\partial^2 U_j}{\partial x_i^2}\right)$$

$$\Rightarrow \quad U_i\frac{\partial^2 U_j}{\partial x_i \partial x_j} + \frac{\partial U_i}{\partial x_j}\frac{\partial U_j}{\partial x_i} = -\frac{1}{\rho}\frac{\partial^2 P}{\partial x_j^2}$$

$$\frac{\partial^2 P}{\partial x_j^2} = -\rho\frac{\partial U_i}{\partial x_j}\frac{\partial U_j}{\partial x_i}$$

Through divergence free condition, pressure uniquely determined by velocity field, independent of flow history.

Vorticity Equation

Turbulent flows are rotational, hence $\boldsymbol{\omega} = \nabla \times \boldsymbol{U} \neq 0$. Take the curl of momentum equation

$$\nabla \times \left(\frac{\partial \boldsymbol{U}}{\partial t} + \boldsymbol{U} \cdot \nabla \boldsymbol{U} = -\frac{1}{\rho} \nabla P + \nu \nabla \cdot \nabla \boldsymbol{U} \right)$$

$$\frac{D\boldsymbol{\omega}}{Dt} = \frac{\partial \boldsymbol{\omega}}{\partial t} + \boldsymbol{U} \cdot \nabla \boldsymbol{\omega} = \nu \nabla \cdot \nabla \boldsymbol{\omega} + \underbrace{\boldsymbol{\omega} \cdot \nabla \boldsymbol{U}}_{\text{vortex stretching}}$$

Remarks:

- Vortex stretching only in 3D
- Vortex stretching essential for transfer of turbulent kinetic energy transfer among different scales

  $\longrightarrow$ Turbulence always three-dimensional